The Regular Number of Adjacent Line Graph of a Ferrers Graph

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Abstract

The regular number $r(G)$ of G is the minimum number of subsets into which the edge set of G is partitioned so that the subgraph induced by each subset is regular. In this article, we investigate the regular numberr_{AL} (G) of Adjacent line graph of a Ferrers graph.

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1.Introduction

In this article, we study a special kind of graph, namely Ferrers graph, which has practical applications in communication networks. The Ferrers relation was introduced for the first time [8] and has been utilized for various purposes across extensive scientific fields. The relation was employed with concept lattices in formal concept analysis [8]. Some graphs associated with the relation were again linked to concept lattices [8]. A kind of partition is presented as Ferrers diagrams. Preference modeling structures were constructed using a generalized version of the relation in Social Choice Theory. In this study, we introduce a new graph class called Ferrers-esque graphs, defined by the relation. We provide a characterization of the class to determine whether an arbitrary simple graph is in the class or not – in other words, whether the graph is a Ferrers-esque graph. 'Ferrers' is used as an abbreviation for 'Ferrers-esque graph.

A simple graph G is a Ferrers graph if for all distinct $x, y, z, w \in V(G)$ if $xy \in E(G)$ and $zw \in E(G)$ then either $xw \in E(G)$ or $yz \in E(G)$. Since $xy \in E(G) \Leftrightarrow yx \in E(G)$ holds for all simple graphs, the definition of Ferrers graph must be extended to if $xy \in E(G)$ and $zw \in E(G)$ then either $xw \in E(G)$ or $yz \in E(G)$ or $yw \in E(G)$ or $xz \in E(G)$. Graphs which do not satisfy the above conditions are classified as non-Ferrers graphs. The graph fails to have at least two disjoint edges to satisfy the Ferrers condition, and this class of graphs is classified as infringe-Ferrers graphs. For example, $K_n, n \geq 4$ is a Ferrers graph and K_n, where $n = 1, 2, 3$ are infringe Ferrers graphs. The path graph P_4 is a Ferrers graph but P_5 is a non-Ferrers graph. The cycle graphs C_4 and $C_5$are Ferrers graphs but C_n, $n \geq 6$ are non-Ferrers graphs. Additionally, C_3 is an infringe Ferrers graph.

Several authors independently discovered the concept of the line graph, each assigning various names to it. In their work [1], Bagga, Beineke, and Varma introduced the idea of super line graphs. The identification of the adjacent line graph was credited to B. Vasudevan and
K.Vimala in 2018.

Let $G = (V, E)$ represent a graph with $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$.The edge graph or line graph of G is denoted by $L(G)$ and has the vertex set E. Two vertices e_i and e_j are adjacent in $L(G)$ if and only if the corresponding edges e_i and e_j of G are adjacent in G. That is, in the line graph $L(G)$ each vertex corresponds to a pair of adjacent vertices in G. Also, in this chapter, we study the concept of Ferrers graph in the adjacent line graph, wherein each vertex corresponds to a pair of adjacent edges in G.

Let $G = (V, E)$ be a simple graph with atleast one pair of adjacent edges. The adjacent line graph of G, denoted by $AL(G)$, is a graph with the vertex set $V_{AL} = \left\{\frac{v_{ij}}{e_i}, e_j \text{are adjacent in} G\right\}$ and two vertices v_{ij}, v_{kl} are adjacent in $AL(G)$, if and only if, either the edges e_i and e_k or e_i and e_l or e_j and e_k or e_j and e_l are adjacent in G.

Moreover, in this paper, we explore a new parameter known as the regular number of a graph. The definition of the regular number was introduced by V.R. Kuli, B.Janakiram, and Radha R. Iyer [6]. The regular number of a graph G, denoted as $r(G)$, is the minimum number of subsets into which the edge set of G should be partitioned so that the subgraph induced by each subset is regular. For every complete graph K_n, where $n \geq 2$, the regular number is one. Similarly, for every cycle graph C_n, where $n \geq 3$, the regular number is one. However, for any path graph P_n, where $n \geq 3$, the regular number, denoted as $r(G)$, is equal to two.This article also introduces the concept of the regular number for the adjacent line graphs of a Ferrers graph.

Definition 1.1.[9]. The *double star graph* $B(m, n)$ is obtained from K_2 by attaching m edges in one vertex and n edges in the other vertex.

Theorem 1.2.[10]. The adjacent line graph of the cycle C_n ($n \geq 5$) is 4-regular.

Theorem 1.3.[10]. The adjacent line graph of the star $K_{1,n}$ is thecomplete graph K_{nc_2}.

Theorem 1.4. [7]. For any graph G, $r(G) = 1$ if and only if G is regular.

Theorem 1.5.[7]. For any tree G, $r(G) = \Delta(G)$.

Theorem 1.6.[3]. Let T be a tree. If T is a Ferrers tree if and only if T has two internal vertices.

Theorem 1.7. For any Ferrers tree T with $n \geq 5$ vertices, the regular number $r(T) \geq 3$.

Theorem 1.8. For any Ferrers tree T with $n > 4$ vertices, $r_L(T) = 2$.

Theorem1.9.[5]Forapath P_n, $AL(P_n)$is aFerrersgraph ifandonlyif $n$is either6or7.

Remark 1.10.[5]If G is a n-pan graph (n ≥ 3), then $AL(G)$ is non complete if it contains exactly three vertices of degree 6, four vertices of degree 5, and the remaining n−5 vertices are of degree4.

Theorem 1.11.[5]. If $G = W_n$ where $n \geq 4$, then $AL(G) = k_p$, where $p = \frac{(n-1)(n+4)}{2}$.

Theorem 1.12.[11]. The adjacent line graph of a cycle, $AL(C_n)$ is a complete graph iff n is either 4 or 5.

Theorem 1.13. The adjacent line graph of a Ferrers tree with $n \geq 5$, which is not a path, is a complete graph if one internal vertex of G has exactly 1 leaf and another internal vertex has atleast 2 leaves.

Theorem 1.14. Let T be a Ferrers tree with two internal vertices such that one is adjacent to one leaf and the other is adjacent to more than one leaf. Then each of the following assertions are equivalent.

i) $AL(T)$ is a complete graph
ii) $diam(AL(T)) = rad(AL(T)) = 1$
iii) $r(AL(T)) = 1$

2. Main Results

In this section, we study the Regular number of Adjacent line graph of a Ferrers graph.

Theorem 2.1. Let $G = W_n$ where $n \geq 4$. Then $r_{AL}(W_n) = 1$

Proof: Consider the adjacent line graph of a wheel graph $G = W_n$ where $n \geq 4$. To prove that $r_{AL}(G) = 1$. By Lemma 1.11, $AL(G)$ is a complete graph. Therefore, by Theorem 1.4, $r_{AL}(G) = 1$.

Theorem 2.2. Let T be a tree with 5 vertices except path. Then$r_{AL}(T) = 1$.

Proof: Let T be a tree with 5 vertices which is not a path. To prove $r_{AL}(T) = 1$.By Theorem 1.13, $AL(T)$ is a complete graph. By Theorem 1.4, $r_{AL}(T) = 1$.

Theorem 2.3. For a path P_6, $r_{AL}(P_6) = 2$.

Proof: Consider a path graph P_6. By Theorem 1.9, $AL(P_6)$ is a Ferrers graph. To prove that, $r_{AL}(P_6) = 2$. Let $v_1, v_2, v_3, v_4, v_5 \text{ and } v_6$ be the vertices of P_6 and e_1, e_2, e_3, e_4, e_5 be the edges of P_6. By the definition of the adjacent line graph $v_{12}, v_{23}, v_{34}, v_{45}$ are the vertices of $AL(P_6)$ and its edge set is given by $v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{45}, v_{12}v_{34}, v_{45}v_{25}$. Now weConsider the followingpartitions. $F_1 = \{v_{12}v_{23}, v_{23}v_{45}, v_{45}v_{34}, v_{34}v_{12}\}$and $F_2 = \{v_{23}v_{34}\}$, and these are the two minimum regular partitions of $AL(P_6)$. Thus, $r_{AL}(P_6) = \{|F_1, F_2|\} = 2$.

Theorem 2.4. For a path P_7 ,$r_{AL}(P_7) = 3$.

Proof: Consider a path P_7. It is clear that, $AL(P_7)$ is a Ferrers graph. To prove that $r_{AL}(P_7) = 3$. Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the vertices of P_7. By the definition of the adjacent line graph, $v_{12}, v_{23}, v_{34}, v_{45}$ and v_{56} are the vertices of $AL(P_7)$ and its edge set is given by$v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{45}, v_{45}v_{56}, v_{12}v_{34}, v_{56}v_{34}, v_{45}v_{23}$. Now we consider the following partitions. $F_1 = \{v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{12}\}$, $F_2 = \{v_{34}v_{45}, v_{45}v_{56}, v_{56}v_{34}\}$, $F_3 = \{v_{23}v_{45}\}$. Clearly, these are the minimum regular partitions of $AL(P_7)$.Thus, $r_{AL}(P_7) = |\{F_1, F_2, F_3\}| = 3$

Theorem 2.5. Prove that $r_{AL}(C_n) = 1$

Proof: Consider a cycle C_n. To prove that $r_{AL}(C_n) = 1$.

Case (i) $n \geq 5$

By Theorem 1.2, $AL(C_n)$ is 4-regular. Also, By Theorem 1.4, $r_{AL}(C_n) = 1$, where $n \geq 5$.

Case (ii) $n = 4$

By Theorem 1.12, $AL(C_4)$ is a complete graph. Therefore, $r_{AL}(C_4) = 1$

Case (iii) $n = 3$

Clearly we observe that, $AL(C_3)$ is K_3. Hence by Theorem 1.4, $r_{AL}(C_3) = 1$.

Theorem 2.6. For any complete graph K_n,$r_{AL}(K_n) = 1$

Proof. Consider a Complete graph K_n. By Theorem 1.4, $r_{AL}(K_n) = 1$.

Theorem 2.7. For any Star graph $K_{1,n}$, where $n \geq 4$, $r_{AL}(K_{1,n}) = 1$.

Proof. Consider star graph $K_{1,n}$, where $n \geq 4$. By Theorem 1.3, $AL(K_{1,n})$ is complete. By Theorem 1.4, $r_{AL}(K_{1,n}) = 1$.

Theorem 2.8. For any Ferrers tree T with $n \geq 5$ vertices, except in the case of a path where one internal vertex has one leaf and other have more than one leaves, then $\delta(T) = r_{AL}(T) < \Delta(T)$.

Proof. By Theorem 1.14, $AL(T)$ is a complete graph. Hence $r_{AL}(T) = 1$. Since G is a Ferrers tree, $\delta(T) = 1$. Also, by Theorem 1.5, $r(T) = \Delta(T) > 1$. Therefore, $\delta(T) = r_{AL}(T) < \Delta(T)$.

Theorem 2.9. For any Ferrers tree T with exactly 5 vertices, $r_{AL}(T) < r(T)$

Proof: Since T is a Ferrers tree with 5 vertices, by Theorem 1.14, $AL(T)$ is a complete graph. Hence, $r_{AL}(T) = 1$. Also, by Theorem 1.7, $r(T) \geq 3$. Therefore, $r_{AL}(T) < r(T)$.

Theorem 2.10. For any Ferrers tree T, $1 \leq r_{AL}(T) \leq 3$

Proof. Consider a Ferrers tree T. To prove $1 \leq r_{AL}(T) \leq 3$. We consider the following cases:

Case (i) T is a Ferrers tree with 5 vertices.

By Theorem 1.6, T has two internal vertices that have exactly one leaf. By Theorem 1.14, $AL(T)$ is a complete graph. Hence, $r_{AL}(T) = 1$.Otherwise, $r_{AL}(T) > 1$.

Case (ii) T is a Ferrers tree with $n \geq 5$ vertices.

Subcase (i)T is a Ferrers tree with $n \geq 5$ vertices and one internal vertex contains exactly one leaf. By Theorem 1.14, $AL(T)$ is a complete graph. Hence, $r_{AL}(T) = 1$.

Subcase (ii). T is a Ferrers tree with $n > 5$ vertices and the two internal vertices contain atleast two leaves.

By Remark 1.10, $AL(T)$ is not a complete graph. By Theorem 1.6, T has two internal vertices. By the definition of the adjacent line graph, the edges in T are converted to the vertices in $AL(T)$. Clearly, all the leaves are incident to the internal vertices. Hence, the maximum number of vertices are adjacent. Now, we find the minimum degree number and we form the regular subgraph(F_1) with the same degree. Also, the remaining edges form as regular subgraphs with the minimum number of degrees. In both the cases, the total number of subgraphs is ≤ 3. Hence, $r_{AL}(T) \leq 3$.

Subcase (iii)T is a Ferrers tree with $n > 5$ vertices and the two internal vertices have the same number of leaves which is ≥ 2.

Since T is a Ferrers tree. Let e_i be the bridge incident to all the leaves. By definition 1.1, the bridge $e_i$is combined with maximum pair of adjacent edges.

Theorem 2.11. For any Ferrers tree T with $n > 5$, $r_{AL}(T) \leq r(T)$.

Proof. Consider T is a Ferrers tree with $n > 5$ vertices. By Theorem 2.10, $1 \leq r_{AL}(G) \leq 3$. Also, by Theorem 1.5, $r(G) = \Delta(G)$. Moreover, by Theorem 1.7, $r(T) \geq 3$. Therefore, $r_{AL}(T) \leq 3 \leq r(T)$. Hence $r_{AL}(T) \leq r(T)$.

Theorem 2.12. For any Ferrers tree T with $n > 5$ vertices, $r_L(T) < r_{AL}(T) \leq r(T)$

Proof. Let T be a Ferrers tree with $n > 5$ vertices. By Theorem 2.8, $r_{AL}(T) \leq r(T)$. Also, by Theorem 1.8 ,$r_L(T) = 2$. Moreover, by Theorem 2.4, $r_{AL}(T) = 3$. Thus, $r_L(T) < r_{AL}(T) \leq r(T)$.

Remark 2.13. When $n = 5$ in Theorem 2.12, $r_{AL}(T) < r_L(T) < r(T)$.

For, Let T be a Ferrers tree with $n = 5$ vertices. By Theorem 1.14, $AL(T)$ is a complete graph and hence $r_{AL}(T) = 1$. By Theorem 1.8, $r_L(T) = 2$. Moreover, by Theorem 1.7, $r(T) \geq 3$.

Therefore, $r_{AL}(T) < r_L(T) \leq r(T)$.

Theorem 2.14. For any Ferrer tree T with $n \geq 5$ vertices, $\delta(T) \leq r_{AL}(T) \leq \Delta(T)$.

Proof. Let Tbe a Ferrers tree. By Theorem 1.6, T has two internal vertices. Since every tree T has pendant vertices, $\delta(T) = 1 \quad \rightarrow(1)$

Case (i). T is a Ferrers tree with $n \geq 5$ vertices and $AL(T)$ is a complete graph.

Since $AL(T)$ is a complete graph, by Theorem 1.4, $r_{AL}(T) = 1 \rightarrow \quad (2)$

From (1) and (2) we get, $\delta(T) = r_{AL}(T) \quad \rightarrow \quad (3)$

Otherwise $\delta(T) < r_{AL}(T)$ → (4)

From (3) and (4) we get, $\delta(T) \leq r_{AL}(T)$ → (A)

Next, we prove that, $r_{AL}(T) \leq \Delta(T)$.

Case (ii). *T* is a Ferrers tree with *n*≥ 5 vertices and $AL(T)$ is not a complete graph.

Since $AL(T)$ is not a complete graph, some vertices in $AL(T)$ have different degrees. Also, the maximum degree of the Ferrers tree is atleast 3, as T contains *n*≥ 5 vertices. First, we form the subgraph with the degree is same as the minimum number of degrees or the minimum number of degrees minus 1. Then, the maximum number of edges is covered in the subgraph. Also, we form the remaining edges from the other subgraphs. Also we form the remaining edges form the other subgraphs. By Theorem 2.10,

$$1 \leq r_{AL}(T) \leq 3 \quad \rightarrow \quad (5)$$

Also, we know that, in a Ferrers tree, the minimum degree of the internal vertices is atleast 3. Hence, $\Delta(G) \geq 3$. By Theorem 1.5.5, $r(G) = \Delta(G) \geq 3$ → (6)

From (5) & (6), We get, $1 \leq r_{AL}(T) \leq \Delta(T)$ → (B)

From (A) and (B) we get, $\delta(T) \leq r_{AL}(T) \leq \Delta(T)$

3. Conclusion

In this article, we found the regular number of Adjacent line graph of a Ferrers graph.

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