Modelling of Sturm-Liouville's Problem via Deformable Derivative

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ABSTRACT*,* This paper presents new results based on a recently proposed deformable derivative [12]. We formulate the Sturm–Liouville problem using the deformable derivative and derive several significant results, including the Rayleigh quotient, criteria for functional dependence and independence via the Wronskian in the deformable derivative framework, and the characterization of the eigenvalues of the deformable-derivative Sturm–Liouville problem.

Keywords – Sturm-Liouville's problems, Deformable derivative, Generalized derivative

1. INTRODUCTION

The slight difference between ordinary calculus and fractional may be remarked while modeling a physical phenomenon. It depends on two parameters first one is the time instant and another important parameter is prior time history which is called memoryless property, and because of the second reasonable property a fractional calculus modeling is achieved. The mentioned key point attracts researchers to contribute to developing efficient techniques regarding the solution of such equations with accuracy. However, advanced calculus has become an emerged mathematics in the last 20 years which is nothing but the generalized concepts of usual calculus and therefore explores its burning properties. It is a subject of the inherent curiosity of the mathematicians which extends an ordinary derivative to an FD. The efforts of researchers led us to several prospective and associated definitions with related properties. The theory includes popular definitions like Riemann-Liouville operators, Caputo operator, Mittag-Leffler function, Erdeleyi–Kober operators, Grunwald-Letnicov operator, Weyl operators, etc. In the last three decades, the excitement of researchers in this subject has been observed in the direction of its applications to the field of engineering and technology like fluid flow, electronic networks, and statistics, etc.[2, 10, 11].

We are interested to demonstrate the related Sturm-Liouville's problem [1] and consequently acquire the individual outcomes. However, a few explanatory investigations have been finished by various researchers. In these investigations, a portion of the results of the Sturm-Liouville issues is reached out to someones with Riemann-Liouville and Caputo and Riesz fractional derivatives. These outcomes incorporate symmetry and fulfillment of eigenfunctions and countability of the eigenvalues.

For brief history of fractional calculus and further developments may search in [9, 8, 7] . Recent applications of fractional derivatives in modelling in emerging areas of research may site in [6].

Presently, at the glance of simplicity and applications, we proceed for the purpose, with the following de formable derivative [4, 3, 5].

2. PRELMINARIES

Definition 2.1. Consider $h: [0, \infty) \to R$ and $\beta \epsilon [0\ , 1], s > 0$ define

(1) $$D^{\beta} h(s) = \lim_{\delta \to 0} \frac{(1+\delta(1-\beta)h(s+\delta\beta)-h(s))}{\delta}$$

$for\ all\ 0 \leq \beta \leq 1$. If the above list exists, the symbol $D^{\beta}[h(s)]$ denotes the deformable derivative of h and is said to be β-differentiable at the point $s \in (a, b)$. If h is Deformable differentiable in$(0,1)$, then

$$D^{\beta} h(0) = \lim_{S \to 0+} D^{\beta} h(s)$$

The function h is known differentiable in deformable derivative sense, if it is differentiable

(2.1) $D^{\beta} h = (1 - \beta)h + \beta Dh$.

Definition 2.2 Let h be continuous on$[a, b]$. Then the deformable Integral is defined as:

(2.2) $I_a^{\beta} h(s) = \frac{1}{\beta} e^{\frac{(\beta-1)}{\beta}s} \int_a^s e^{\frac{(1-\beta)}{\beta}x} . h(x) dx, \quad \beta \epsilon (0, 1]$

Remark 2.3 The above definition is evident for $\beta = 0, 1$ i.e. $D^0[h(s)] = h(s)$ (function itself) and $D[h(s)] = h'(s)$ (first ordinary derivative).

3. FRACTIONAL ORDER STURM-LIOUVILLE'S EIGENVALUE PROBLEM

If $h(s)$ is two times deformable derivative differentiable function, we consider the following generalized Sturm-Liouville's problem

(3.1)

$$D^{\beta}\big(p(s) D^{\beta} h\big) + q(s)h = -\lambda \eta(s) h\ ; \frac{1}{2} < \beta \leq 1, \qquad a < s < b$$

s.t. $h'(a) + kh(a) = 0, h'(b) + lh(b) = 0\ ; l > 0$

where p is β-differentiable and q, η are positive and continuous functions of s in $[a, b]$.

We define operator

(3.2) $K(h, \beta) = D^{\beta}\big(p(s) D^{\beta} h\big) + q(s)h$

So that (4) can be written as

$$K(h, \beta) = -\lambda \eta(s) h.$$

Let g and f be deformable derivative function then β Wronskian is defined by the following

$$W_{\beta}(g, f) = \begin{vmatrix} f & g \\ D^{\beta} f & D^{\beta} g \end{vmatrix}$$

$$= \beta(fg' - gf')$$
$$= \beta W(f, g).$$

Where $W(f, g)$ denotes ordinary Wronskian. Let h_1, h_2 be two times continuously β-differentiable functions on

$[a, b]$ then

(3.3) $[h_2K(h_1, \beta) - h_1K(h_2, \beta)] = D^\beta[p(s)(h_2D^\beta h_1 - h_1D^\beta h_2)] + \beta(1-\beta)p(s)W$

where W denotes the ordinary Wronskian.

Proof.

$$\begin{aligned}[h_2K(h_1, \beta) - h_1K(h_2, \beta)] &= h_2D^\beta(pD^\beta h_1) - h_1D^\beta(pD^\beta h_2)\\ &= p[h_2D^\beta D^\beta h_1 - h_1D^\beta D^\beta h_2] + \beta p'(h_2D^\beta h_1 - h_1D^\beta h_2)\\ &= p[\beta^2W' + 2\beta(1-\beta)W] + \beta p'(s)(\beta W)\\ &= \beta^2D[p(s)W] + 2\beta(1-\beta)p(s)W\end{aligned}$$

and

$$D^\beta[p(s)(h_2D^\beta h_1 - h_1D^\beta h_2)] = \beta D^\beta(p(s)W)$$

$$= \beta((1-\beta) + \beta D)p(s)W$$

$$= \beta^2D[p(s)W] + \beta(1-\beta)p(s)W$$

With Deformable Integral operator

(3.4) $\int_a^b[h_2K(h_1, \beta) - h_1K(h_2, \beta)]\, d'_\beta s = [p(s)(h_2D^\beta h_1 - h_1D^\beta h_2)]_a^b - \int_a^b p(s)W\, d_\beta s$

If h_1, h_2 are Deformable differentiable on $[a, b]$ and holds the condition of (4) then

(3.5) $[p(s)(h_2D^\beta h_1 - h_1D^\beta h_2)]_a^b = 0.$

Proof. We have

$$p(b)[h_2(b)D^\beta h_1(b) - h_1(b)D^\beta h_2(b)] - p(a)[h_2(a)D^\beta h_1(a) - h_1(a)D^\beta h_2(a)]$$

(3.6) $= \beta p(b)[h_2h'_1 - h_1h'_2]_{s=b} - \beta p(a)[h_2h'_1 - h_1h'_2]_{s=a}$

Now using boundary condition $h'(b) = -lh(b)$ we have

$$\beta p(b)[h_2h'_1 - h_1h'_2]_{s=b} = \beta p(b)[h_2(-lh_1) - h_1(-lh_2)]_{s=b} = 0$$

Similarly we can show

$$\beta p(a)[h_2h'_1 - h_1h'_2]_{t=a} = 0$$

Using other boundary condition.

If $h'(a) + kh(a) = 0, h'(b) + lh(b) = 0$ with $k, l > 0$ then

$[W(h_1, h_2)]_a^b = [W_\beta(h_1, h_2)] = 0$

and therefore,

(3.7) $\int_a^b[h_2K(h_1, \beta) - h_1K(h_2, \beta)]\, d_\beta s = 0$

Eigen functions of (4) with distinct eigenvalues are Deformable orthogonal along with $\eta(s)$.

Proof. Let λ_1, λ_2 be two distinct eigenvalues corresponding to eigen functions h_1, h_2 respectively then

$K(h_1, \beta) = \lambda_1\eta(s)h_1$

and $K(h_2, \beta) = \lambda_2\eta(s)h_2$

so that

(3.8) $h_1K(h_2, \beta) - h_2K(h_1, \beta) = (\lambda_2 - \lambda_1)\eta(s)h_1h_2$

And

$$(\lambda_2 - \lambda_1)\int_a^b \eta(s)\, h_1 h_2 d_\beta s = \int_a^b [h_1 K(h_2,\beta) - h_2 K(h_1,\beta)]\, d_\beta s = 0$$

Now $\lambda_1 \neq \lambda_2$ implies

$\int_a^b \eta(s)\, h_1 h_2 d_\beta s = 0$

All the eigenvalues of (4) are real.

Proof. $K(\bar{h},\beta) = D^\beta\big(p(s), \overline{D^\beta h}\big) + q(s)\bar{h} = -\lambda\eta(s)\bar{h}$

and boundary conditions

$\bar{h}'(a) + k\bar{h}'(a) = 0$

and $\bar{h}'(b) + k\bar{h}(b) = 0$

$$(\bar{\lambda} - \lambda)\int_a^b \eta(s)\, |\lambda|^2 d_\beta s = \int_a^b \big[hK(\bar{h},\beta) - \bar{h}\, K(h,\beta)\big]\, d_\beta s$$

$$= \big[p(s)\big(h_2 D^\beta h_1 - h_1 D^\beta h_2\big)\big]_a^b$$

$$= 0.$$

Therefore $\lambda = \bar{\lambda}$.

Eigenvalues of (4) are linearly dependent (simple).

Proof. Since $W_\beta(h_1,h_2) = 0$ and hence the result.

Let h_1, h_2 be two linearly independent solutions of the problem (4) then

$p(t)W_\beta(h_1,h_2) = ce^{-\frac{(1-\beta)s}{\beta}}$

where c is constant.

Proof.

(3.9) $D^\beta W_\beta(h_1,h_2) = h_1 D^\beta D^\beta h_2 - h_2 D^\beta D^\beta h_1 + \beta\big(h_1' D^\beta h_2 - h_2' D^\beta h_1\big)$ and

(3.10) $D^\beta D^\beta h = -\frac{1}{p(s)}\big(\beta p'(s) D^\beta h + (\lambda\eta(s) + q(s))h\big)$

Now

$$D^\beta W_\beta(h_1,h_2) = \frac{\beta p'(s)}{p(t)}\big(h_2 D^\beta h_1 - h_1 D^\beta h_2\big) + \beta\big(h_1' D^\beta h_2 - h_2' D^\beta h_1\big)$$

$= \frac{\beta p'(s)}{p(s)}\big(h_2 D^\beta h_1 - h_1 D^\beta h_2\big) + \beta(1-\beta)(h_1' h_2 - h_2' h_1)$

$$= -\frac{\beta p'(s)}{p(s)} W_\beta(h_1,h_2) - \beta(1-\beta)W(h_1,h_2)$$

$$= -\frac{\beta p'(s)}{p(s)} W_\beta(h_1,h_2) - (1-\beta)W_\beta(h_1,h_2)$$

$$= -\frac{W_\beta(h_1,h_2)}{p(s)} D^\beta p(s)$$

And $p(s)W_\beta(h_1,h_2) = ce^{-\frac{(1-\beta)s}{\beta}}$

Eigenvalues λ of (4) must satisfy

(3.11) $$\lambda = \frac{-\int_a^b q(s)h(s)d_\beta s - \left[p(s)D^\beta h(s) + ce^{-\frac{(1-\beta)s}{\beta}}\right]_a^b}{\int_a^b \eta(s)h(s)d_\beta s}$$

Proof. Integrating (4)

(3.12) $$\int_a^b D^\beta\big(p(s)D^\beta h\big)\, d_\beta s + \int_a^b q(s)\, hd_\beta s \; = -\lambda \int_a^b \eta(s)h\, d_\beta s$$

(3.13) $$\int_a^b q(s)\; h(s)d_\beta s + \left[p(s)D^\beta h(s) + ce^{-\frac{(1-\beta)s}{\beta}}\right]_a^b = -\lambda \int_a^b \eta(s)h(s)\, d_\beta s$$

And therefore,

$$\lambda = \frac{-\int_a^b q(s)h(s)d_\beta\, s - \left[p(s)D^\beta h(s) + ce^{-\frac{(1-\beta)s}{\beta}}\right]_a^b}{\int_a^b \eta(s)h(s)\, d_\beta s}$$

4. CONCLUSION

We have modelled a fractional Sturm-Liouville eigenvalue problem in Deformable derivative derivative sense. We have shown that as regular Sturm-Liouville's problem, the eigenvalues are real and simple and the eigenfunctions are orthogonal. We also established the fractional Wronskian result for any two linearly independent solutions of the problem.

5. CONFLICT-OF-INTEREST

The author declares no conflicts of interest.

6. ACKNOWLEDGEMENT AND FUNDING

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7. DATA AVAILABILITY

This manuscript has no associated data.

8. ETHICAL CONDUCT

Not applicable

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