Extreme Non-Split Geodesic Graphs

Shobha[1,*],Venkatesh S H[2,*],Tejaswini K M[3,*],
L.S.Chitra[4,] and Venkanagouda M Goudar[5,*]
[*]Department of Mathematics, Sri Siddhartha Institute of Technology,
A Constituent College of Sri Siddhartha Academy of Higher Education,
Tumkur,
Karnataka, India.
[4]Department of Mathematics, Adhichunchanagiri Institute of Technology
Chikkamagaluru, Karnataka, India
Email[1]:shobhashree30@gmail.com, Email[2]:sh1.venkatesh@gmail.com,
Email[3]:tejaswini.ssit@gmail.com, Email[4]: chitrals.ait@gmail.com
Email[5]:vmgouda@gmail.com

Abstract

Here we study the new concept of extreme non-split geodesic graph in $G = (V(G), E(G))$. The geodetic set $S \subseteq V(G)$ is called a non-split geodetic set, if the induced sub graph $< V(G) - S >$ is connected. The cardinality of a non-split geodetic set which is minimum is the non-split geodetic number $g_{ns}(G)$. The extreme order $ex(G)$ is the number of extreme vertices in a graph G. The connected graph G is an extreme non-split geodesic graph if $g_{ns}(G) = ex(G)$. Here we determine the extreme non-split geodesic graph G of $radius\ r$ and $diameter\ d$ with $g_{ns}(G) = k$ for the integers r and $d, k > 1$ with $r < d \leq 2r$. Also we proved $ex(G) = a$ and $g_{ns}(G) = b$ for the integers a, $b \geq 2$ and $0 \leq a \leq b$ in a connected graph G.

Keywords: Corona product, Extreme geodesic graph, Extreme vertex, Non-split geodetic number.

Subject Classification: AMS-05C12, 05C76.

1 Introduction

In this paper, the graph $G = (V, E)$ is a simple, finite and connected with $V(G)$ the vertex set containing $n \geq 3$ vertices and the edge set $E(G) \subseteq V \times V$. Let d(u, v) denote the distance between any two points u, v in G. An u-v path of length d(u,v) is a u-v geodesic. This concept was introduced in [1]. The interval $I[u, v]$ containing all the points lying on some $u - v$ geodesic in G and $S \subseteq V(G)$, $I[S] = \cup_{u,v \in S} I[u, v]$. A set S of vertices is a geodetic set if $I[S] = V(G)$. The cardinality of a geodetic set which is minimum is the geodetic number $g(G)$.

The geodetic set S is said to be a non-split geodetic set in G, if $< V(G) - S >$ is connected. The cardinality of a non-split geodetic set which is minimum is the non-split geodetic number $g_{ns}(G)$. The concept of non-split geodetic number was introduced and studied in [5].

A vertex v in G is called an extreme vertex if the subgraph induced by its neighbours is a complete graph. Each extreme vertex belongs to a geodetic set. A graph G is an extreme geodesic graph if $g(G) = ex(G)$, that is, if every vertex lies on a u, v geodesic for some pair u, v of extreme vertices. The concept of extreme geodesic graphs was first introduced in [2]. Contraction of pair of vertices is an operation where two or more vertices in a graph are merged into a single new vertex. Various concepts inspired by geodetic sets are introduced in [3, 4] .

Theorem 1.1. [5] The tree T with n vertices and k end vertices, $g_{ns}(T) = k$.

2 Extreme non-split geodesic graphs

In this paper, we have introduced extreme non-split geodesic graph. The graph G is said to be an extreme non-split geodesic graph if the non-split geodetic number $g_{ns}(G)$ is equal to the number of extreme vertices in a graph G.

In the graph G_1, the set $S = \{v_2, v_4, v_6\}$ is a geodetic set with minimum cardinality and $< V(G_1) - S >$ is connected. So the set S forms a non-split geodetic set with $g_{ns}(G_1) = 3$. Also the only three extreme vertices of G_1 are the vertices in S. Thus, $g_{ns}(G_1) = ex(G_1) = 3$. Therefore, the connected graph G_1 is an extreme non-split geodesic graph.

In a graph G_2, the geodetic set $S = \{v_2, v_4, v_5\}$ is a non-split geodetic set so $g_{ns}(G_2) = 3$ and the two extreme vertices are v_2, v_4 in G_2, so $ex(G_2) = 2$. Clearly, $g_{ns}(G_2) \neq ex(G_2)$. Therefore, the graph G_2 is not an extreme non-split geodesic graph.

The graph G_3 in Figure 2.1, any two antipodal vertices form minimum non-split geodetic set and there is no extreme vertex in G_3. So $ex(G_3) \neq g_{ns}(G_3)$. Thus, the graph G_3 is not an extreme non-split geodesic graph.

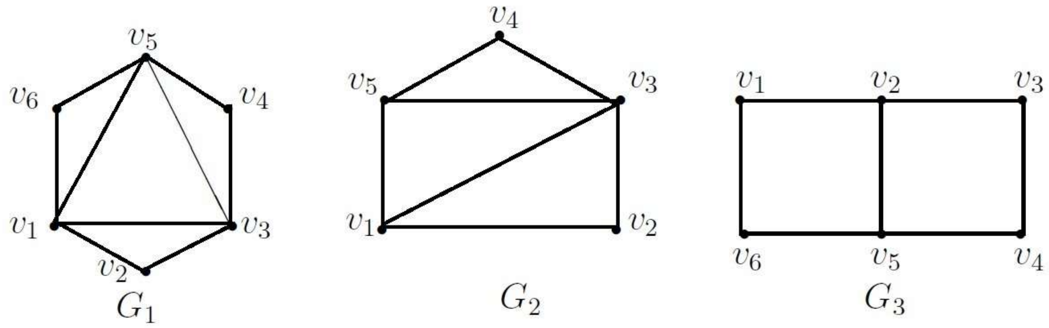

Figure 2.1

Observation 2.1 There is no extreme vertex in a cycle $C_n, n \geq 4$, complete bipartite graph $K_{m,n}, 2 \leq m \leq n$ so that the graphs C_n, $K_{m,n}$ are not an extreme non-split geodesic graphs.
Observation 2.2 In a complete graph $G = K_n$, there is no non-split geodetic set but G is an extreme geodesic graph. So $G = K_n$ is not an extreme non-split geodesic graph.
Observation 2.3 For any tree T with n vertices, k pendant vertices and $n - k \geq 1$, $g_{ns}(T) = ex(T) = k$. Therefore, T is an extreme non-split geodesic graph.

Theorem 2.4 The extreme non-split geodesic graph G of order n with $ex(G) = n$ does not exists.
Proof. Assume that G is an extreme non-split geodesic graph of order n with $ex(G) = n = g_{ns}(G)$. Then all tthe vertices of G are extreme and forms a geodetic set X. Since $< V(G) - X >$ is an order-zero graph, there is no non-split geodetic set so $g_{ns}(G) < ex(G) = n$. This contradicts to the assumption that G is an extreme non-split geodesic graph. Thus, the extreme non-split geodesic graph of order n with $ex(G) = n$ does not exists.

Theorem 2.5 For any integer k such that $2 \leq k < n$, an extreme non-split geodesic graph G of order n with $g_{ns}(G) = k$ exists.
Proof. Consider the graph $G = T$, a tree in Figure 2.2 having n vertices and k extreme vertices where $2 \leq k \leq n - 1$ so that $ex(G) = k$. By the Theorem 1.1, the non-split geodetic number $g_{ns}(G) = k$. Since $g_{ns}(G) = ex(G) = k$ the tree $G = T$ is an extreme non-split geodesic graph.

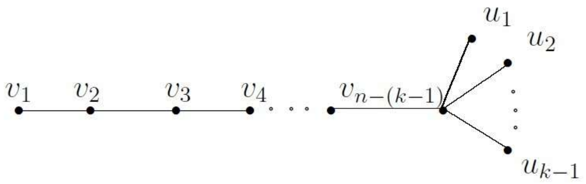

Figure 2.2: G

Theorem 2.6 The extreme non-split geodesic graph G of radius r and diameter d with $g_{ns}(G) = k$ exists for the integers r and $d, k > 1$ with $r < d \leq 2r$.

Proof. Suppose that $r = 1$, $d = 2$. Consider a cycle C_4 containing four vertices u_1, u_2, u_3, u_4. The resultant graph G is formed by joining u_1 and u_3 vertices by an edge and also joining the pendant vertices $\{v_i\}$ where $1 \leq i \leq (k-2)$, to a vertex u_3 in C_4 is as shown in Figure 2.3. The set of all extreme vertices in G is $S = \{v_1, v_2, \ldots, v_{k-2}, u_2, u_4\}$ forms a geodetic set. Since $< V(G) - S >$ is connected, the set S of k extreme vertices is a non-split geodetic set. Therefore, $g_{ns}(G) = k = ex(G)$ and the graph G is an extreme non-split geodesic graph.

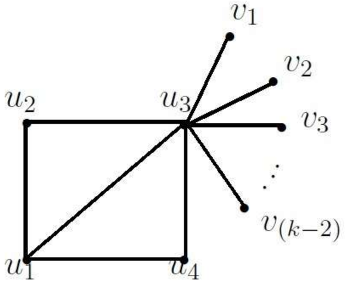

Figure 2.3: G

Suppose that $1 < r < d$. Consider a cycle C_{2r} containing $v_1, v_2, \ldots, v_{2r}, v_1$ vertices and a path P_{d-r} of length $d - r$ with vertices u_i, $0 \leq i \leq d - r$. The graph G formed by contraction of pair of vertices v_1 in C_{2r}, u_0 in P_{d-r} and joining the two vertices v_r, v_{r+2} in C_{2r} by an edge also by adding $k - 2$ pendant vertices $\{w_i\}$ where $1 \leq i \leq (k-2)$ to the vertex u_{d-r-1} in P_{d-r} is shown in Figure 2.4. The set $S = \{w_1, w_2, \ldots, w_{k-2}, u_{d-r}, v_{r+1}\}$ containing k extreme vertices in a graph G forms a geodetic set and $< V(G) - S >$ is connected. Clearly, the set S is also a non-split geodetic set. Since $g_{ns}(G) = k = ex(G)$, the connected graph G is an extreme non-split geodesic graph.

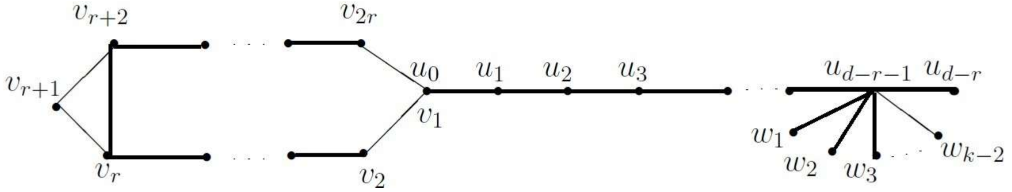

Figure 2.4: G

Theorem 2.7 For three integers $n, d \geq 3$ and $k \geq 2$ with $d, k < n$ and also $n - d - k + 1 \geq 0$, the extreme non-split geodesic graph G exists with $g_{ns}(G) = k$ of order n .

Proof. Consider the path P_{d+1} having vertices $u_1, u_2, \ldots, u_{d+1}$ of diameter d. The graph G is formed by joining $n - d - 1$ new vertices of two different sets $\{w_i\}$ where $1 \leq i \leq n - d - k + 1$ and $\{v_j\}$, $1 \leq j \leq k - 2$ to the path P_{d+1}. For all the three vertices u_1, u_2 *and* u_3, join each $\{w_i\}$ where $1 \leq i \leq n - d - k + 1$ and for the vertex u_d, join each pendant vertices $\{v_j\}$ where $1 \leq j \leq k - 2$ in a path P_{d+1}. Also all vertices of $\{w_i\}$ where $1 \leq i \leq n - d - k + 1$ induces a complete graph on $n - d - k + 1$ vertices. The resultant graph G of order n is as shown in

Figure 2.5. The set $S = \{u_1, u_{d+1}\} \cup \{v_j\}$ where $1 \leq j \leq k-2$ containing k vertices in G forms a geodetic set with $< V(G) - S >$ is connected. These k extreme vertices forms a non-split geodetic set. Thus, the graph G is an extreme non-split geodesic graph with $g_{ns}(G) = k$.

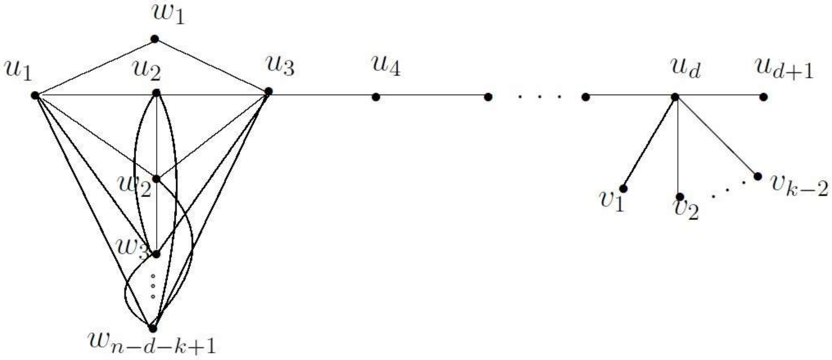

Figure 2.5: G

Theorem 2.8 For three integers $\mathrm{k} \geq 2$ and $\mathrm{n}, \mathrm{d} \geq 3$ with $\mathrm{d}, \mathrm{k} < \mathrm{n}$ also $\mathrm{n} - \mathrm{d} - \mathrm{k} + 1 \geq 0$. The connected graph G exists with $\mathrm{g}_{\mathrm{ns}}(\mathrm{G}) = \mathrm{k}$ which is not extreme non-split geodesic graph having diameter d of order n.

Proof. Consider the path P_{d+1} having vertices $u_1, u_2, \ldots, u_{d+1}$ of diameter d. The graph G is formed by adding $n-d-1$ new vertices of two different sets $\{w_i\}$ where $1 \leq i \leq n-d-k+1$ and $\{v_j\}$, $1 \leq j \leq k-2$ to the path P_{d+1}. For all the three vertices u_1, u_2, u_3 join each $\{w_i\}$ where $2 \leq i \leq n-d-k+1$ and to the vertex u_d, join each pendant vertices $\{v_j\}$ where $1 \leq j \leq k-2$. Also join w_1 to the vertices u_1, u_3 in P_{d+1} and all vertices of $\{w_i\}$ where $2 \leq i \leq n-d-k+1$ induces the complete graph on $n-d-k+1$ vertices is as shown in Figure 2.6. The set $S = \{u_{d+1}, v_1, v_2, \ldots, v_{k-2}\}$ of $k-1$ extreme vertices in G is not a geodetic set. Clearly $ex(G) = k-1$ and $g(G) > k-1$. Consider the set $S' = S \cup \{u_1\}$ is a non-split geodetic set with k vertices and so $g_{ns}(G) = k \neq ex(G)$. Therefore, the graph G is not an extreme non-split geodesic graph.

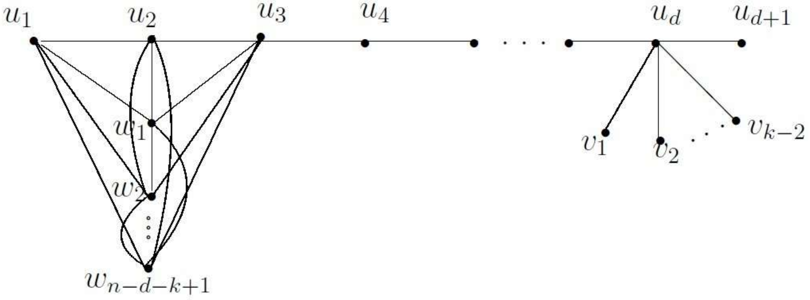

Figure 2.6: G

Theorem 2.9 For two integers $\mathrm{a}, \mathrm{b} \geq 2$ and also $0 \leq \mathrm{a} \leq \mathrm{b}$, the connected graph G exists with $\mathrm{ex}(\mathrm{G}) = \mathrm{a}$ and $\mathrm{g}_{\mathrm{ns}}(\mathrm{G}) = \mathrm{b}$.

Proof. Case 1: $0 = a < b$. Consider the path P_d having vertices u_0, u_1, u_2. The graph G is formed by adding $\{w_i\}$, $1 \leq i \leq (b-2)$ vertices to both u_0 and u_2 in P_d is as shown in Figure 2.7. The set $S = \{u_0, u_2, w_1, w_2, \ldots, w_{b-2}\}$ is a non-split geodetic set and there is no extreme vertex in G. Therefore, $ex(G) = 0 = a < b = g_{ns}(G)$.

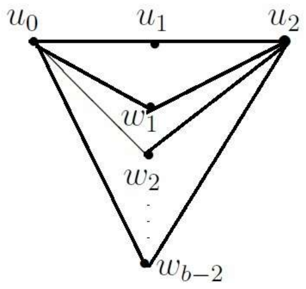

Figure 2.7: G

Case 2: $0 < a = b$. It follows from Theorem 2.6.

Case 3: $0 < a < b$.

If $a < b$ with $b = a + 1$, the graph G is formed by joining the pendant vertices $\{l_1, l_2, \ldots, l_a\}$ to both the vertices v_{r+1} and v_{r+2} in a cycle C_n containing $V(C_n) = \{v_1, v_2, \ldots, v_{r+1}, v_{r+2}, \ldots, v_{2r+1}, v_1\}$ is as shown in Figure 2.8. In G, the set $S = \{l_1, l_2, \ldots, l_a\}$ containing extreme vertices so $ex(G) = a$. Clearly, this is not a non-split geodetic set so $g_{ns}(G) > a$. The set with $a + 1$ elements $S \cup \{v_1\}$ is the minimum unique non-split geodetic set and so $g_{ns}(G) = a + 1 = b$.

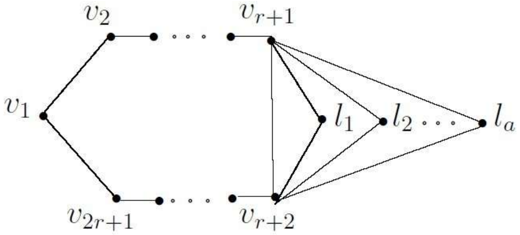

Figure 2.8: G

If $a < b$ where $b = a + 2$. Let G be a graph obtained by joining the new vertex v to the graph in Figure 2.8 and join this vertex to both v_2, v_{2r+1}. The set $S = \{l_1, l_2, \ldots, l_a\}$ containing extreme vertices in G so, $ex(G) = a$ and $S \cup \{v_1\}$ containing $(a + 1)$ vertices is not a geodetic set in G. Clearly, $g_{ns}(G) > a + 1$. The set $S' = S \cup \{v_1, v\}$ forms a non-split set and so $g_{ns}(G) = b$ where $b = a + 2$.

Suppose $a < b$ where $b \geq a + 3$. Consider a cycle C_n containing $V(C_n) = \{v_1, v_2, \ldots, v_{r+1}, \ldots, v_{2r+1}, v_1\}$ and join the vertex v_1 to both v_3, v_{2r}. Add $'a'$ number of pendant vertices $\{l_1, l_2, \ldots, l_a\}$ and join these to both v_{r+1} and v_{r+2} in C_n. The resultant graph G is produced by joining all $(b - a - 2)$ vertices w_j where $1 \leq j \leq b - a - 2$ to both v_2, v_{2r+1} is as shown in Figure 2.9. In G, the set $S = \{l_1, l_2, \ldots, l_a\}$ of extreme vertices is not a geodetic set so $g(G) > a$. The set $S_1 = S \cup \{v_2, v_{2r+1}\}$ is a geodetic set and $< V(G) - S_1 >$ is disconnected. Clearly, S_1 is not a non-split geodetic set and $g_{ns}(G) > a + 2$. Consider the set $S_2 = S_1 \cup \{w_1, w_2, \ldots, w_{b-a-2}\}$ containing b vertices forms the non-split geodetic set which is minimum. Hence, $g_{ns}(G) = b$.

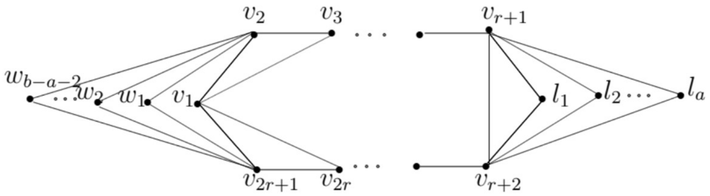

Figure 2.9: G

Theorem 2.10 The graph G' is formed by joining r-pendant vertices $v_i \notin G$ where $1 \leq i \leq r$ to each vertex in a graph $G = C_n$, where C_n is a cycle , $n \geq 4$. Then the extreme non-split geodesic graph G' exists with $g_{ns}(G') = k$

Proof. The graph G' formed by joining r −pendant vertices $v_i \notin G$ where $1 \leq i \leq r$ to each vertex in $G = C_n$ of order n. The set S in G' containing extreme vertices of cardinality $nr = k$ is an unique non-split geodetic set. Therefore, $g_{ns}(G') = ex(G') = nr = k$. Thus, G' is extreme non-split geodesic graph.

3 Extreme non-split geodesic graph and corona graphs

In this section, we determine the extreme non-split geodesic graph for certain classes of corona graph.

Definition 3.1 A graph formed by taking a single copy of G with $|V(G)| = n$ copies of H. By joining i^{th} vertex of G with each vertex in the i^{th} copy of H, then it is said to be a corona $G \circ H$ of two graphs G and H.

Theorem 3.2 For the graphs $G = P_n, T, C_n, K_n, K_{m,n}$ and $H = K_1$, the extreme non-split geodesic graph $G \circ H$ exists with $g_{ns}(G \circ H) = ex(G \circ H) = k$ where $k = n$ for $G = P_n, T, C_n, K_n$ and $k = m + n$ for $G = K_{m,n}$.

Proof. (i) If $G = P_n$ and $H = K_1$ then by the definition of corona graphs $G \circ H = P_n \circ K_1$ becomes a caterpillar tree with $2n$ vertices and $n = k$ end vertices. By Observation 2.4, $(P_n \circ K_1)$ is extreme non-split geodesic graph with $g_{ns}(G \circ H) = ex(G \circ H) = k$.

(ii) By the definition, $T \circ K_1$ is a tree with $2n$ vertices and $n = k$ pendant vertices. By Observation 2.4, the tree $T \circ K_1$ is an extreme non-split geodesic graph with $g_{ns}(T \circ K_1) = ex(T \circ K_1) = k$.

(iii) By the definition, $C_n \circ K_1$ is the connected graph containing $n = k$ end vertices. The set S containing these end vertices is the minimum non-split geodetic set also $n = k$ end vertices are all extreme vertices. Thus, $g_{ns}(C_n \circ K_1) = k = ex(C_n \circ K_1)$. Therefore, $(C_n \circ K_1)$ is an extreme non-split geodesic graph.

(iv) By the definition, $K_n \circ K_1$ is a connected graph containing k extreme vertices. The set S containing these k vertices forms the non-split geodetic set. Therefore, $(K_n \circ K_1)$ is an extreme non-split geodesic graph.

(v) By the definition, $K_{m,n} \circ K_1$ is a connected graph containing $2(m + n)$ vertices and the set S with $m + n = k$ end vertices. The set S with these $(m + n) = k$ end vertices in $K_{m,n} \circ K_1$ is the minimum non-split geodetic set so $g_{ns}(K_{m,n} \circ K_1) = k$ and all the end vertices are extreme so $ex(K_{m,n} \circ K_1) = k$. Therefore, $(K_{m,n} \circ K_1)$ is an extreme non-split geodesic graph.

4 Extreme non-split geodesic graph of some special graphs

4.1 Friendship graph

Theorem 4.1 The friendship graph F_n is an extreme non-split geodesic graph with $g_{ns}(F_n) = k$.

Proof. The friendship graph F_n is a planar graph of order $2n + 1$ and size $3n$ formed by joining n copies of a cycle C_3 with a common vertex x is as shown in Figure 4.1. In the graph F_n, all $2n = k$ vertices adjacent to a common vertex x are extreme vertices forms a non-split geodetic set. Clearly, $g_{ns}(F_n) = ex(F_n) = k$. Thus, F_n is an extreme non-split geodesic graph.

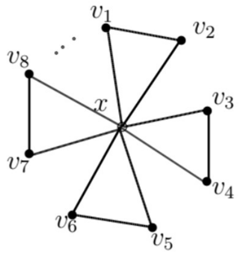

Figure 4.1: Friendship graph

4.2 Triangular snake graph

Theorem 4.2 The triangular snake graph $\mathrm{TS_n}$ is an extreme non-split geodesic graph with $\mathrm{g_{ns}(G) = k}$.

Proof. Triangular snake graph graph TS_n containing n odd number vertices formed from a path by replacing every edge from a cycle C_3. In TS_n, $\left\lceil \frac{n+2}{2} \right\rceil = k$ vertices are extreme and forms a non-split geodetic set is as shown in Figure 4.2. Clearly, $g_{ns}(TS_n) = ex(TS_n) = k$. Thus, TS_n is an extreme non-split geodesic graph.

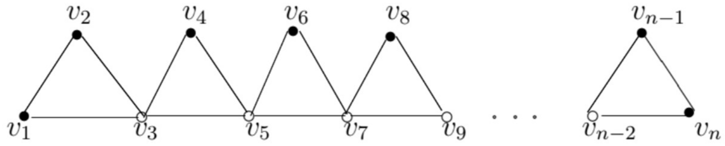

Figure 4.2: Triangular snake graph

Theorem 4.3 The alternate triangular snake graph $\mathrm{A(TS_n)}$ is an extreme non-split geodesic graph with $\mathrm{g_{ns}(A(TS_n)) = k}$.

Proof. The alternate triangular snake graph $A(TS_n)$ is formed by replacing the alternate edges by a cycle C_3 in a path P_n. The graph $A(TS_n)$ containing $2n - 1$ edges and $\frac{3n}{2}$ vertices with $\frac{n}{2} + 2$ extreme vertices is as shown in Figure 4.3. The set with $\left(\frac{n}{2} + 2 \right) = k$ extreme vertices forms a non-split geodetic set. Clearly, $g_{ns}(A(TS_n)) = ex(A(TS_n))$. Thus, $(A(TS_n))$ is an extreme non-split geodesic graph.

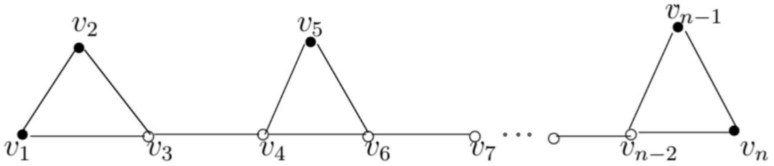

Figure 4.3: Alternate triangular snake graph

4.3 Windmill graph

Theorem 4.4 The windmill graph $\mathrm{W_d(k, n)}$ is an extreme non-split geodesic graph.

Proof. The windmill graph $W_d(k,n)$ of order $[(k-1)n+1]$ and size $\left[\dfrac{nk(k-1)}{2}\right]$ is formed by joining n copies of a complete graph K_k with a common vertex.
For $k=3$, the windmill graph $W_d(3,n)$ is a friendship graph and the result follows from the Theorem 4.1. Consider $k>3$ and $n>1$. In this graph, all $(k-1)n$ vertices other than common vertex are extreme vertices and forms an unique non-split geodetic set. Therefore $W_d(k,n)$ is an extreme non-split geodesic graph.

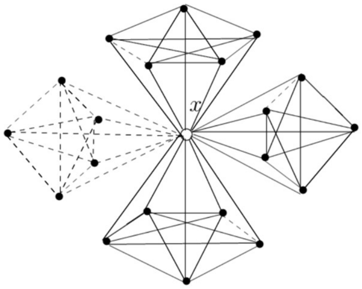

Figure 4.4: Windmill graph

4.4 Lollipop graph

Theorem 4.5 The (m,n)-lollipop graph is an extreme non-split geodesic graph.
Proof. The (m,n)-lollipop graph is formed by joining a vertex x_m in a complete graph K_m containing the vertex set $\{x_1,x_2,\dots,x_{m-1},x_m\}$ to an initial vertex v_1 in a path P_n containing the vertex set $\{v_1,v_2,\dots,v_n\}$ with an edge is shown in Figure 4.5. The set $\{x_1,x_2,\dots,x_{m-1},v_n\}$ of m extreme vertices forms non-split geodetic set so $g_{ns}(G)=ex(G)$. Thus, the (m,n)-lollipop graph is an extreme non-split geodesic graph.

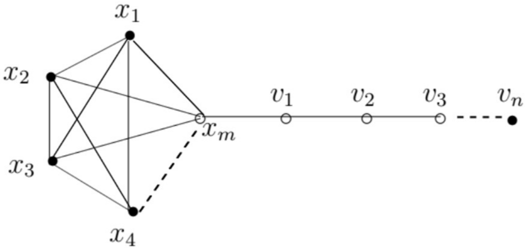

Figure 4.5: Lollipop graph

4.5 Torch graph

Theorem 4.6 The Torch graph $\mathrm{O_n}$ is an extreme non-split geodesic graph with $\mathrm{g_{ns}(O_n)=n}$.
Proof. The torch graph of order $n>3$ is O_n has $n+4$ vertices and $2n+3$ edges is as shown in Figure 4.6. The set of extreme vertices containing n elements $S=\{v_{n-1},v_{n+4},v_{n+2},v_2,v_3,\dots,v_{n-2}\}$ forms a geodetic set and $<V(G)-S>$ is connected. Thus, the set S of n extreme vertices forms non-split geodetic set so, $g_{ns}(O_n)=ex(O_n)=n$. Thus, the torch graph is an extreme non-split geodesic graph.

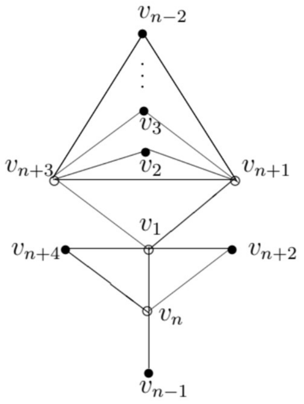

Figure 4.6: Torch graph for $n > 3$

4.6 Helm graph

Theorem 4.7 The Helm graph H_n is an extreme non-split geodesic graph with $g_{ns}(H_n) = n$.
Proof. The helm graph $G = H_n$ is an undirected graph of order $2n + 1$ and size $3n$ formed from an n-wheel graph by adding a pendant edge at each vertex of the cycle is shown in Figure 4.7. The set of extreme vertices $S = \{w'_1, w'_2, w'_3, \ldots, w'_n\}$ forms a geodetic set and $< V(H_n) - S >$ is connected. Thus the set S of n extreme vertices forms non-split geodetic set so $g_{ns}(H_n) = ex(H_n) = n$. Thus, H_n is an extreme non-split geodesic graph.

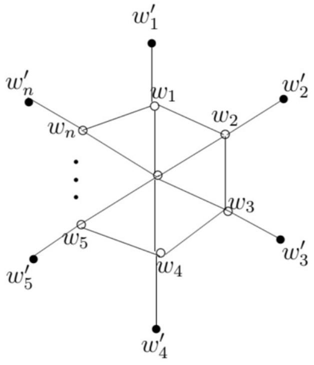

Figure 4.7: Helm graph

5 Conclusion

In this paper, we obtained the results on extreme non-split geodesic graph and also some general properties of corona product of graphs. Further, we studied the results on integers, some special graphs.

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